

ON THE G -INVARIANT MODULES

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ABSTRACT. Let G be a reductive group acting on a path algebra kQ as automorphisms. We assume that G admits a graded polynomial representation theory, and the action is polynomial. We describe the quiver Q_G of the smash product algebra $kQ \# k[M_G]^*$, where M_G is the associated algebraic monoid of G . We use Q_G -representations to construct G -invariant representations of Q . As an application, we construct algebraic semi-invariants on the quiver representation spaces from those G -invariant representations.

INTRODUCTION

Let k be a field of characteristic 0, and A be a finite-dimensional k -algebra with a finite group G acting as automorphisms. Then we can form the skew group algebra $AG := A \# k[G]$, which is a well-studied subject (e.g., [13]). AG and A have the same representation type and global dimension. If the algebra is the path algebra of a finite quiver Q , and the action permutes primitive idempotents and stabilizes the arrow span kQ_1 , then the quiver Q_G of kQG can be explicitly described [1, 9] (see Section 1.1).

A natural question is that if G is a reductive group acting rationally on A as automorphisms, what is a good analogue of the skew group algebra? One natural answer can be replacing the group algebra by the Hopf algebra $k[G]$, and forming the smash product $A \# k[G]^*$. However, the dual coordinate algebra $k[G]^*$ is not semisimple, and quite complicated in general. To describe the quiver of $kQ \# k[G]^*$ is a rather difficult task. So we consider the coordinate (bi)algebra of the *associated monoid* $k[M_G]$ as an alternative. If G admits a *graded polynomial representation theory* (Definition 2.1), then $k[M_G]^*$ is semisimple. So the price is that we need to restrict to a special class of reductive groups and require the action to be polynomial. Then we can explicitly describe the quiver Q_G of $kQ[M_G]^* := kQ \# k[M_G]^*$. The quiver is possibly an infinite quiver, but each connected component is still finite-dimensional (Proposition 3.4). Theorem 3.3 is our first main result. The proof is similar to that in [1].

Let us come back to the finite group action. The action of G on A induces an action of G on the category of (left) A -modules. We write this induced action in the exponential form, that is, M^g is the module M with the action of A twisted by g :

$$am = (g^{-1}a)m.$$

An A -module M is called *G -invariant* if $M^g \cong M$ for any $g \in G$. The restriction of an AG -module M is a G -invariant A -module. The converse is almost true

2010 *Mathematics Subject Classification*. Primary 16S40; Secondary 16S35, 16G20, 13A50.

Key words and phrases. Quiver Representation, G -invariant Representation, Hopf Action, Skew Group Algebra, Smash Product, Schur Algebra, Morita Equivalence, Semi-invariant, Tensor Invariants, Littlewood-Richardson, Kronecker Coefficient.

(Lemma 1.2). Those kQ -modules admitting a kQG -module structure are our main interest. In fact, we only need something weaker called *proj-coherently G -invariant* (Definition 1.3). They contain all exceptional representations of Q (Observation 1.4). To construct such kQ -modules, we need to concretely describe the Morita equivalence functor $kQ_G\text{-mod} \rightarrow kQG\text{-mod}$ composed with the restriction functor $kQG\text{-mod} \rightarrow kQ\text{-mod}$. This can be done as long as we can compute all idempotents of the group algebra $k[G]$ (see Section 1.2).

All above about finite group actions have analogue for our $kQ[M_G]^*$. However, in this case Q_G is possibly an infinite quiver, so it is quite impossible to completely describe the above functor. So we fix some connected component Q_c of Q_G , then we can describe the analogous functor $kQ_c\text{-mod} \rightarrow kQ\text{-mod}$, provided we can compute all idempotents of some homogenous subalgebra S_c of $k[M_G]^*$ depending on Q_c . Such subalgebra S_c is a finite direct product of *Schur algebras* of G .

Our motivation comes from constructing algebraic semi-invariants on the quiver representation spaces. For some dimension vector α , let $\text{Rep}_\alpha(Q)$ be the space of all α -dimensional representations of Q . The product of general linear group $\text{GL}_\alpha := \prod_{v \in Q_0} \text{GL}_{\alpha(v)}$ acts on $\text{Rep}_\alpha(Q)$ by the natural base change. In [14], Schofield introduced for each representation $N \in \text{Rep}_\beta(Q)$ with $\langle \alpha, \beta \rangle_Q = 0$, a semi-invariant function $c_N \in k[\text{Rep}_\alpha(Q)]$ for the above action. Here $\langle -, - \rangle_Q$ is the Euler form of Q . In fact, c_N 's span the space of all semi-invariants of weight $-\langle -, \beta \rangle_Q$ over the base field k [2, 15].

The action of G on kQ induces G -actions on all representation spaces of Q . An easy observation is that if N is proj-coherently G -invariant, then c_N is also semi-invariant under G -action. This observation allows us to construct new semi-invariants for the $\text{GL}_\alpha \times G$ -action on $k[\text{Rep}_\alpha(Q)]$. We are particularly interested in the setting of n -arrow Kronecker quivers K_n , where $G = \text{GL}_n$ acting on the space of arrows. The (α_1, α_2) -dimensional representation space of K_n can be identified with the (tri-)tensor space $U^* \otimes V \otimes W^*$, where $\dim(U, V, W) = (\alpha_1, \alpha_2, n)$. To illustrate our method, we construct several such semi-invariants in Proposition 4.2, 4.3, 4.4, and 4.5. Proposition 4.2 may be well-known, but we believe that the rest are new.

We hope to find the dimension of the linear span of semi-invariants of form c_N , where N is coherently G -invariant of fixed dimension. Theorem 4.7 converts this problem to a similar problem on the quiver Q_c . As we will see, when Q_c is simple, the dimension can be easily calculated.

Notations and Conventions. Our vectors are exclusively row vectors. If an arrow of a quiver is denoted by a lowercase letter, then we use the same capital letter for its linear map of a representation. For direct sum of n copy of M , we write nM instead of the traditional $M^{\oplus n}$. Unadorned Hom and \otimes are all over the base field k , and the superscript $*$ is the trivial dual.

1. FINITE GROUP ACTION

Let k be a field of characteristic 0, and G be a finite group acting on a k -algebra A as automorphisms. The group algebra $k[G]$ is a Hopf algebra with counit, comultiplication, and antipode defined by the linear extension of

$$\epsilon(g) = 1, \quad \Delta(g) = g \otimes g, \quad S(g) = g^{-1}.$$

In this way, A obtains a $k[G]$ -module algebra structure.

Definition 1.1. Let B be a bialgebra. A (left) B -module algebra A is an algebra which is a (left) module over B such that for any $b \in B, a, a' \in A$,

$$b1_A = \epsilon(b)1_A, \quad \text{and} \quad b \cdot (aa') = \sum (b_{(0)} \cdot a)(b_{(1)} \cdot a').$$

The *smash product algebra* $A \# B$ is the vector space $A \otimes B$ with the product

$$(a \otimes b)(a' \otimes b') := \sum a(b_{(0)} \cdot a') \otimes b_{(1)} b'.$$

When $B = k[G]$ is a group algebra, we may abuse of notation writing a for $a \otimes 1_G$ and b for $1_A \otimes b$. In this context, $a \otimes b$ can be written as ab , and thus $A \# k[G]$ may be denoted by AG .

The action of G on A induces an action of G on the category of (left) A -modules. We write this induced action in the exponential form, that is, M^g is the module M with the action of A twisted by g :

$$am = (g^{-1}a)m.$$

For morphisms $f \in \text{Hom}_A(M, N)$, we check that the following defines a morphism $f^g \in \text{Hom}_A(M^g, N^g)$

$$f^g(m) = f(m).$$

If M is an A -module then $(AG) \otimes_A M$ is isomorphic as an A -module to $\bigoplus_{g \in G} M^g$, where the action of G permutes the factors.

We observe that an AG -module M is an A -module which is also a G -module, and such that

$$(1.1) \quad g(am) = (ga)(gm).$$

Lemma 1.2. *An A -module M is an AG -module if and only if there is a family of isomorphisms $\{i_g : M \rightarrow M^g\}_{g \in G}$ satisfying $i_h^g i_g = i_{hg}$ for any $g, h \in G$.*

Proof. If M is an AG -module, then (1.1) says that the assignment $m \mapsto g^{-1}m$ defines an isomorphism $i_g : M \rightarrow M^g$. Conversely, if we have a family of isomorphisms $\{i_g : M \rightarrow M^g\}_{g \in G}$ satisfying $i_h^g i_g = i_{hg}$ for any $g, h \in G$, then we can endow M with a G -module structure as follows. Note that M and M^g have the same underlying vector spaces on which $i_h^g = i_h$, so we can define a G -action on M satisfying (1.1) by $g(m) = i_g(m)$. \square

Definition 1.3. An A -module M is called *G -invariant* if $M^g \cong M$ for any $g \in G$. It is called *proj-coherently G -invariant* if there is a family of isomorphisms $\{i_g : M \rightarrow M^g\}_{g \in G}$ satisfying that $\forall g, h \in G, \exists c \in k^*$ such that $i_h^g i_g = c \cdot i_{hg}$. It is called *coherently G -invariant* if it admits a AG -module structure. A G -invariant A -module is called *(coherently) G -indecomposable* if it is not a direct sum of two (coherently) G -invariant modules.

For our main application on invariant theory, we are more interested in (proj-)coherently G -invariant modules. However, we do not know any example where M is G -invariant but not coherently G -invariant. When G is cyclic and A a path algebra, Gabriel [5] proved that they are equivalent.

Observation 1.4. *Let $A = kQ$ be the path algebra of a finite quiver Q .*

- (1) *A rigid representation of Q is G -invariant.*
- (2) *A G -invariant Schur representation of Q is proj-coherently G -invariant.*

- (3) If the cohomology group $H^2(G; k^*)$ vanishes, then proj-coherent is equivalent to coherent.

Proof. By definition M is rigid if $\text{Ext}_Q^1(M, M) = 0$. So the orbit of M is dense in its representation space, which is irreducible. But M^g is rigid as well, so they have to be in the same orbit.

By definition, M is Schur if $\text{Hom}_Q(M, M) = k$. So the statement follows from the definition.

If $H^2(G; k^*) = 0$, then every projective representation $G \rightarrow \text{GL}_\alpha/k^*$ lifts to $G \rightarrow \text{GL}_\alpha$. So we can modify each i_g by some scalar factor such that $i_h^g i_g = i_{hg}$. \square

Definition 1.5. A dimension vector α of Q is called a G -root if there is an α -dimensional coherently G -indecomposable representation. It is called a *strong* G -root if there is an indecomposable coherently G -invariant representation.

When G is cyclic, all G -roots can be described in terms of the root system of associated valued quiver [7]. The following lemma is well-known.

Lemma 1.6. For any finite-dimensional algebra A , AG and A have the same global dimension and representation type.

1.1. Description for Q_G . By Lemma 1.6, kQG is Morita equivalent to some hereditary algebra kQ_G . There are algorithms to find the quiver Q_G if the action permutes primitive idempotents and stabilizes the arrow span kQ_1 . Let us recall the methods in [1, 9].

Let \tilde{Q}_0 be a set of representatives of class of Q_0 under the action of G . For $u \in Q_0$, let O_u be the orbit of u and G_u be the subgroup of G stabilizing e_u .

For $(u, v) \in \tilde{Q}_0 \times \tilde{Q}_0$, G acts diagonally on the product of the orbits $O_u \times O_v$. A set of representatives of the classes of this action will be denoted by O_{uv} . We define $R_{uv} := kQ(u, v)$ to be the vector space spanned by the arrows from u to v . We regard R_{uv} as a right $k[G_{uv}] := k[G_u \cap G_v]$ -module by restricting the action of G .

Let $\text{irr}(G)$ denote the set of all irreducible representations of G . The vertex set of Q_G is

$$\bigcup_{v \in \tilde{Q}_0} \{u\} \times \text{irr}(G_u).$$

The arrow set from (u, ρ) to (v, σ) is a basis of

$$\bigoplus_{(u', v') \in O_{uv}} \text{Hom}_{k[G_{u'v'}]}(V_\rho, R_{u'v'} \otimes V_\sigma).$$

Here ρ should be understood as a representation of $G_{u'}$ as follows. Let $g_{uu'}$ be such that $g_{uu'}u = u'$, then $\rho(h) = \rho(g_{uu'}^{-1} h g_{uu'})$ for $h \in G_{u'}$. Similar identification makes σ a representation of $G_{v'}$.

The proof uses the following idempotent e of kQG , which will be used later. Let R be the maximal semisimple subalgebra of kQ . Let $e_0 = \sum_{u \in \tilde{Q}_0} e_u \in R \subset RG$. It is not hard to see that $e_0(kQG)e_0$ is Morita equivalent to kQG , and $e_0(RG)e_0 \cong \prod_{u \in \tilde{Q}_0} k[G_u]$. Since each G_u is semi-simple, we can fix for each $u \in \tilde{Q}_0$ and $\rho \in \text{irr}(G_u)$, a primitive idempotent $e_{u\rho}$ of $k[G_u]$ corresponding to ρ . Let

$$e = \sum_{u \in \tilde{Q}_0} \sum_{\rho \in \text{irr}(G_u)} e_{u\rho}.$$

It is proved in [1] that $e(kQG)e$ is a basic algebra Morita equivalent to kQG .

1.2. Functors. Let $A := kQ$ and $B := kQG$. The functor $AG \otimes_A -$ has the restriction functor as its right adjoint. The Morita equivalence functor $e(-)$ has $R_e := \text{Hom}_B(eAG, -)$ as its right adjoint. So the composition $T := e(AG \otimes_A -)$ has a right adjoint $R := \text{res} \circ R_e$. Note that T is exact and preserves projective presentations, and thus R preserves injective presentations. Moreover, both T and R map semisimple modules to semisimple modules [13, Theorem 1.3].

The functor $AG \otimes_A -$ is also right adjoint to the restriction functor [13, Theorem 1.2]. So T also has a left adjoint $L := \text{res} \circ AGe \otimes_B -$. However, in this notes we will exclusively work with the functor R .

Now we have the following diagram of functors

$$\begin{array}{ccc}
 & \text{mod } A \# G & \\
 \begin{array}{c} \nearrow A \# G \otimes_A - \\ \searrow \text{res} \end{array} & & \begin{array}{c} \nwarrow e(-) \\ \searrow R_e \end{array} \\
 \text{mod } A & \xleftarrow{R} & \text{mod } B
 \end{array}$$

By our construction, the functor R sends the simple $S_{u\rho}$ corresponding to the vertex $e_{u\rho}$ to the semisimple representation $\bigoplus_{v \in O_u} \dim(V_\rho) S_v$ of Q . In this way, R induces a linear map $r : K_0(B) \rightarrow K_0(A)$. Since R_e is an equivalence and preserves indecomposables, it follows that

Proposition 1.7. *α is a G -root if and only if there is a root β of Q_G such that $r(\beta) = \alpha$.*

We want to give a concrete description for the functor R . To be more precise, we want to lift R to a map between representation spaces of Q_G and Q . Clearly, such a description relies on the choice of a complete set of primitive orthogonal idempotents of $k[G_u]$ for each $u \in \tilde{Q}_0$. In general, no explicit formula for primitive orthogonal idempotents in a finite group algebra is known. However, in many special cases, for example when the group is a symmetric group, a complete set of primitive orthogonal idempotents is given by the Young symmetrizers (1.2) [6, 9.3].

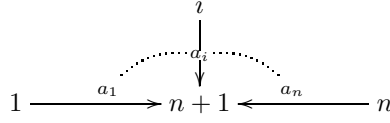
Assume that we have got a complete set I of primitive orthogonal idempotents of $k[G_u]$ for each $u \in \tilde{Q}_0$. By Maschke's Theorem, $k[G_u]$ is a product of matrix algebras $\prod_{\rho \in \text{irr}(G_u)} \text{End}(V_\rho)$. We can compute a standard basis $\{e_{u\rho}^{ij}\}$ of the matrix algebra $\text{End}(V_\rho)$ such that $\{e_{u\rho}^{ii}\} \subset I$ and $e_{u\rho}^{11} = e_{u\rho}$. We identify a basis of $\{e_{u\rho}^{1i} R_{uv} e_{v\sigma}^{j1}\}$ with some arrows from (u, ρ) to (v, σ) , say $\{b_k\}_k$. Now for each $a \in R_{uv}$, $\{e_{u\rho}^{ii} a e_{v\sigma}^{jj}\}$ is a linear combination of b_k 's. Say $e_{u\rho}^{ii} a e_{v\sigma}^{jj} = \sum c_k^{ij} b_k$.

For any $N \in \text{Rep}_\beta(Q_G)$, $M = R(N) \in \text{Rep}_{r(\beta)}(Q)$ is the following representation. The vector space M_u attached to the vertex u is

$$M_u = \bigoplus_{\rho \in \text{irr}(G_u)} d_\rho N_{u\rho}, \quad d_\rho = \dim(V_\rho).$$

Here, each copy of $N_{u\rho}$ corresponds to some $e_{u\rho}^{ii}$. Let us denote such a copy by $N_{u\rho}^i$. The linear map from $N_{u\rho}^i$ to $N_{v\sigma}^j$ is given by substituting the arrows in $\sum_k c_k^{ij} b_k$ by corresponding matrices in N . In particular, we see that such a lifting is an algebraic morphism $\text{Rep}_\beta(Q_G) \rightarrow \text{Rep}_{r(\beta)}(Q)$.

Example 1.8. Let S_n be the n -subspace quiver:

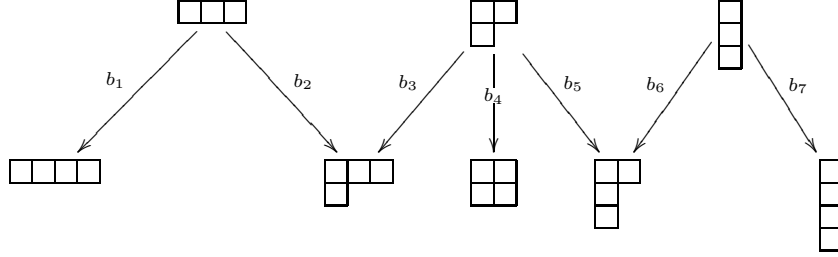


The symmetric group \mathfrak{S}_n acts naturally on S_n . In this way, we get an action of \mathfrak{S}_n on kS_n . There are only two orbits on Q_0 represented by n and $n+1$. The stabilizer G_n and G_{n+1} are \mathfrak{S}_{n-1} and \mathfrak{S}_n respectively. We have only one orbit in $O_n \times O_{n+1}$. The irreducible representations of S_n are indexed by partitions ρ , and primitive idempotents in $\text{End}(V_\rho)$ can be labeled by Young tableaux T of shape ρ :

$$(1.2) \quad e_T = \kappa_\rho^{-1} \sum_{v \in V(T)} \sum_{h \in H(T)} \text{sgn } v \cdot vh.$$

Here, κ_ρ is the hook length of ρ , $V(T), H(T)$ are the vertical and horizontal subgroup corresponding to the Young tableaux T . The number of arrows between (n, ρ) and $(n+1, \sigma)$ is given by the multiplicity of ρ in σ restricted to \mathfrak{S}_{n-1} -module. This is equal to the Littlewood-Richardson coefficients $c_{\rho,1}^\sigma$, which can be computed by the Pieri rule.

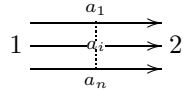
For $n = 4$, we get the following quiver for B



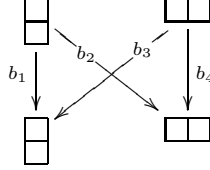
The functor R takes a representation of the above quiver to the following representation of S_4 .

$$\begin{aligned} A_1 &= \begin{pmatrix} B_1 & B_2 & -B_2 & -B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_3 & 0 & B_3 & B_4 & 0 & B_5 & -2B_5 & B_5 & 0 \\ 0 & 0 & B_3 & -B_3 & -B_4 & -B_4 & B_5 & B_5 & -2B_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_6 & B_6 & B_6 & B_7 \end{pmatrix} \\ A_2 &= \begin{pmatrix} B_1 & B_2 & -B_2 & 3B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_3 & 0 & 0 & B_4 & B_4 & B_5 & 3B_5 & 0 & 0 \\ 0 & 0 & B_3 & 0 & -B_4 & 0 & B_5 & 0 & 3B_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_6 & 0 & 0 & -B_7 \end{pmatrix} \\ A_3 &= \begin{pmatrix} B_1 & B_2 & 3B_2 & -B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_3 & 0 & 0 & -B_4 & -B_4 & -3B_5 & -B_5 & 0 & 0 \\ 0 & 0 & 0 & B_3 & 0 & B_4 & 0 & -B_5 & -3B_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -B_6 & 0 & -B_7 \end{pmatrix} \\ A_4 &= \begin{pmatrix} B_1 & -3B_2 & -B_2 & -B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -B_3 & 0 & -B_4 & 0 & 3B_5 & 0 & B_5 & 0 \\ 0 & 0 & 0 & 0 & B_3 & 0 & -B_4 & 0 & 3B_5 & B_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_6 & B_7 \end{pmatrix}. \end{aligned}$$

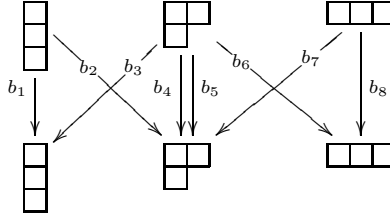
Example 1.9. The symmetric group \mathfrak{S}_n also acts naturally on the n -arrow Kronecker quiver K_n



The \mathfrak{S}_n -representation on arrows decomposes into the standard representation and the trivial representation, So the number of arrows between $(1, \rho)$ and $(2, \sigma)$ is given by $g_{\rho, [n-1, 1]}^\sigma + \delta_{\rho, \sigma}$. Here, $g_{\rho, \pi}^\sigma$ is the Kronecker coefficient defined by $V_\rho \otimes V_\pi = \oplus g_{\rho, \pi}^\sigma V_\sigma$. Readers can verify the following quivers Q_G together with the functor R for $n = 2, 3$.



$$A_1 = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}, A_2 = \begin{pmatrix} B_1 & -B_2 \\ -B_3 & B_4 \end{pmatrix}.$$



$$A_1 = \begin{pmatrix} B_1 & B_2 & B_2 & 0 \\ B_3 & \frac{B_4+B_5}{2} & \frac{B_5-B_4}{2} & B_6 \\ -B_3 & \frac{B_4-B_5}{2} & -\frac{B_4+B_5}{2} & B_6 \\ 0 & B_7 & -B_7 & B_8 \end{pmatrix}, A_2 = \begin{pmatrix} B_1 & B_2 & -2B_2 & 0 \\ 0 & B_4 & 0 & -2B_6 \\ B_3 & \frac{B_5-B_4}{2} & -B_5 & B_6 \\ 0 & -B_7 & 0 & B_8 \end{pmatrix}, A_3 = \begin{pmatrix} B_1 & -2B_2 & \frac{B_2}{2} & 0 \\ -B_3 & B_5 & \frac{B_4-B_5}{2} & B_6 \\ 0 & 0 & -B_4 & -2B_6 \\ 0 & 0 & B_7 & B_8 \end{pmatrix}.$$

2. SCHUR ALGEBRAS OF REDUCTIVE MONOID

In this section, we recall several results from [3]. We keep our assumption that the base field k has characteristic 0. Let M_n be the affine algebraic monoid of $n \times n$ matrices over k . We naturally identify the coordinate algebra $k[M_n]$ with the polynomial algebra $A(n) := k[X]$, where $X = \{x_{ij}\}_{1 \leq i \leq j \leq n}$. The polynomial algebra is graded by the usual monomial degree $A(n) = \bigoplus_{d \geq 0} A(n, d)$. Moreover, $A(n)$ is a bialgebra with coalgebra structure maps Δ, ϵ defined by

$$\Delta(x_{ij}) = \sum_{k=1}^n x_{ik} \otimes x_{kj}, \quad \epsilon(x_{ij}) = \delta_{ij}.$$

Thus each graded piece $A(n, d)$ is a subcoalgebra of $A(n)$. Hence, its linear dual $S(n, d) := A(n, d)^*$ is a finite-dimensional k -algebra, known as the classical Schur algebra.

The coordinate algebra of the general linear group GL_n is the localization of $A(n)$ at the determinant function: $k[GL_n] = k[X, \det(X)^{-1}]$. Let G be a reductive closed subgroup of GL_n . By a polynomial function on G , we mean the restriction to G of a polynomial function in $A(n)$. We denote by $A(G)$ the algebra of polynomial function on G . It inherits the coalgebra structure and the grading from $A(n)$. Its degree d piece $A(G, d)$ is a subcoalgebra. We denote the linear dual of $A(G, d)$ by $S(G, d)$. It is a subalgebra of $S(n, d)$ because $A(G, d)$ is a quotient of $A(n, d)$.

Definition 2.1. We say that G admits a *graded polynomial representation theory* if the sum $\sum_{d \geq 0} A(G, d)$ is direct.

A standard non-example is SL_n because $A(\mathrm{SL}_n, 0) \cap A(\mathrm{SL}_n, n) \neq \emptyset$ due to the equation $\det(X) = 1$. It is not hard to see that if G contains the nonzero scalar matrices cI of GL_n , then G admits a graded polynomial representation theory. This includes, for example GSp_n and GO_n , the groups of symplectic and orthogonal similitudes. Proposition 2.3 provides another criterion.

A finite dimensional (left) $k[G]$ -module V is called rational if for some basis v_1, \dots, v_n of V the corresponding coefficient functions f_{ij} , defined by the equations

$$g \cdot v_i = \sum_{j=1}^n f_{ij}(g) v_j$$

belong to $k[G]$. We then have on V the structure of a right $k[G]$ -comodule via the structure map $\Delta_V : V \rightarrow V \otimes k[G]$, given by $\Delta_V(v_i) = \sum_{j=1}^n v_j \otimes f_{ij}$. It is well-known that there is an equivalence of categories between rational $k[G]$ -module and $k[G]$ -comodules. By a polynomial G -module we mean a vector space V on which G acts linearly with coefficient functions in $A(G)$.

Proposition 2.2. [3, Proposition 1.3, 1.4] *Suppose G admits a graded polynomial representation theory. Every polynomial G -module has a direct sum decomposition by homogeneous polynomial representations. The category of homogeneous polynomial G -modules of degree d is equivalent to the category of $S(G, d)$ -modules.*

We take $M_G = \overline{G}$, the Zariski closure of G in M_n . Then M_G is a closed submonoid of M_n with G as its group of units. M_G is called the associated algebraic monoid of G . Let $I(M_G)$ be the vanishing ideal of M_G in M_n .

Proposition 2.3. [3, Proposition 2.4] *G admits a graded polynomial representation theory if and only if $I(M_G)$ is homogeneous. In this case, we have a coalgebra isomorphism $A(G, d) \cong k[M_G]_d$, so the algebra $S(G, d)$ consists of those elements in $S(n, d)$ vanishing on $I_d(M_G) = A(n, d) \cap I(M_G)$.*

We provide last point of view of $S(G, d)$ from the tensor power representations. Let V be the (n -dimensional) natural M_n -representation. For any $d \in \mathbb{N}$, we have an action of M_n on the d th tensor power of V , by

$$A(v_1 \otimes \cdots \otimes v_d) = Av_1 \otimes \cdots \otimes Av_d.$$

Let ϕ_d be the corresponding representation $M_n \rightarrow \mathrm{End}(V^{\otimes d})$. It was proved by Schur [10] that $S(n, d) = \phi_d(\mathrm{GL}_n) = \phi_d(M_n) = \mathrm{End}_{\mathfrak{S}_d}(V^{\otimes d})$.

Proposition 2.4. [3, Proposition 3.2] *If G admits a graded polynomial representation theory, then*

$$S(G, d) = \phi_d(G) = \phi_d(M_G).$$

It is well-known that the semisimplicity of $\phi_d(G)$ is equivalent to complete reducibility of $V^{\otimes d}$ as G -module. So $\phi_d(G)$ is semisimple if G is reductive. Combined with a monoid analogue of the Peter-Weyl theorem [12, Proposition 13], we have that

Lemma 2.5. *As G -bimodule algebras, $S(G, d) \cong \bigoplus_{\rho} \mathrm{End}(V_{\rho})$, where ρ runs through all irreducible degree d polynomial representations of G . So if G admits a graded polynomial representation theory, then as G -bimodule algebras,*

$$k[M_G]^* \cong \prod_{\rho \in \mathrm{irr}(G)} \mathrm{End}(V_{\rho}),$$

where $\text{irr}(G)$ is the set of all irreducible polynomial representations of G .

Knowing that $S(G, d)$ is semisimple, it is an important problem to determine a complete set of primitive orthogonal idempotents. This can be a very hard problem in general, but for the classical Schur algebras $S(n, d)$, it is treated in [4]. Here is some simple examples, which will be used later.

Recall that the standard monomial basis of $A(n, d)$ is indexed by the *generalized permutations* $\begin{pmatrix} i_1 & i_2 & \dots & i_n \\ j_1 & j_2 & \dots & j_n \end{pmatrix}$. The pairs (i_k, j_k) are arranged in non-decreasing lexicographic order from left to right. In other words, the i 's are arranged in increasing order, and the j 's corresponding to the same i are in increasing order. We denote the corresponding dual basis in $S(n, d)$ by $\xi_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_n}$. A nice combinatorial rule for multiplying such a basis is given in [11].

Example 2.6. Let $A = S(n, 2)$. It has the following complete set of idempotents

$$\begin{aligned} & \left\{ \frac{1}{2}(\xi_{ij}^{ij} - \xi_{ij}^{ji}) \right\}_{1 \leq i < j \leq n} & \begin{array}{|c|} \hline \square \\ \hline \end{array} \\ & \left\{ \xi_{ii}^{ii}, \frac{1}{2}(\xi_{ij}^{ij} + \xi_{ij}^{ji}) \right\}_{1 \leq i < j \leq n} & \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \end{aligned}$$

The right column indicates the corresponding irreducible representations.

Example 2.7. Let $A = S(n, 3)$. It has the following complete set of idempotents

$$\begin{aligned} & \left\{ \frac{1}{6} \sum_{\omega \in \mathfrak{S}_3} \text{sgn}(\omega) \xi_{\omega(ijk)}^{ijk} \right\} & \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \\ & \left\{ \frac{1}{3}(2\xi_{ii}^{ii} - \xi_{ii}^{ji}), \frac{1}{3}(2\xi_{ij}^{ij} - \xi_{ij}^{ji}), \frac{1}{3}(\xi_{ijk}^{ikj} - \xi_{jki}^{ijk} + \xi_{ikj}^{ijk} - \xi_{jik}^{ijk}), \frac{1}{3}(\xi_{ijk}^{ijk} - \xi_{ikj}^{ijk} + \xi_{jik}^{ijk} - \xi_{kij}^{ijk}) \right\} & \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \\ & \left\{ \xi_{iii}^{iii}, \frac{1}{3}(\xi_{ii}^{ij} + \xi_{ii}^{ji}), \frac{1}{3}(\xi_{ij}^{ij} + \xi_{ij}^{ji}), \frac{1}{6} \sum_{\omega \in \mathfrak{S}_3} \xi_{\omega(ijk)}^{ijk} \right\} & \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \end{aligned}$$

where $1 \leq i < j < k \leq n$.

3. REDUCTIVE GROUP ACTION

Definition 3.1. Let B be a k -bialgebra. A (right) B -comodule algebra A is a k -algebra with a right B -comodule structure $\Delta_A : A \rightarrow A \otimes B$. We required Δ_A to be a k -algebra homomorphism. The *smash product algebra* $A \# B^*$ is by definition the vector space $A \otimes B^*$ with multiplication

$$(c \otimes h)(a \otimes f) = \sum a_{(0)} c \otimes (a_{(1)} \cdot h) f.$$

Here $a_{(1)} \cdot h$ is the usual (left) B -action on B^* , that is, $a_{(1)} h(b) = h(ba_{(1)})$.

We observe that a left A -module M , which is also a right B -comodule $\Delta_M : M \rightarrow M \otimes B$ such that

$$\Delta_M(am) = \Delta_A(a) \Delta_M(m)$$

is a left $A \# B^*$ -module, but not vice versa. We may abuse of notation writing a and f for $a \otimes 1_{B^*}$ and $1_A \otimes f$. If $\Delta_A(1) = 1_A \otimes 1_{B^*}$, then we will write af for $a \otimes f$ and AB^* for $A \# B^*$ in this context.

Let G be an infinite connected reductive group over k , and M_G be the associated algebraic monoid. Since G is algebraic, we will only consider rational action of G . In fact, we assume that G acts polynomially as automorphisms on some k -algebra A . Then A becomes a $k[M_G]$ -comodule algebra. As in the finite group case, we

also have a G -action on the category of A -modules. We define (proj-coherently) G -invariant and G -indecomposable module as before.

The group G can be naturally embedded into the dual coordinate algebra $k[M_G]^*$. For every $g \in G$, we define $\epsilon_g \in k[M_G]^*$ as $\epsilon_g(f) = f(g)$. Moreover, the embedding respects actions: $\epsilon_g(m) = \sum \epsilon_g(m_{(1)})m_{(0)} = \sum m_{(1)}(g)m_{(0)} = gm$.

Lemma 3.2. *If M is an $A\#k[M_G]^*$ -module, then $m \mapsto gm$ defines an A -module isomorphism $M \cong M^g$ for all $g \in G$.*

Proof. We need to show for all $g \in G, a \in A, m \in M$ that

$$(1 \otimes \epsilon_g)(a \otimes 1)(m) = g(am) = (ga)(gm).$$

Suppose for all $g \in G, a \in A, m \in M$ that

$$gm = \sum m_1(g)m_0, \text{ and } ga = \sum a_1(g)a_0.$$

Then

$$\begin{aligned} (1 \otimes \epsilon_g \cdot a \otimes 1)(m) &= \sum a_{(0)} \otimes (a_{(1)} \cdot \epsilon_g)(m) \\ &= \sum a_{(0)} (a_{(1)} \cdot \epsilon_g(m_{(1)})) m_{(0)} \\ &= \sum a_{(0)} \epsilon_g(m_{(1)} a_{(1)}) m_{(0)} \\ &= \sum a_{(0)} m_{(1)}(g) a_{(1)}(g) m_{(0)} \\ &= (ga)(gm). \end{aligned}$$

□

Conversely, given a G -invariant A -module M , we assume that for each $g \in G$ we can fix an isomorphism $i_g : M \rightarrow M^g$ such that $i_h^g i_g = i_{hg}$. Then we can define a G -action on M by $g(m) = i_g(m)$. If such an action can be extended to $k[M_G]^*$ (e.g., the action is polynomial), then we get an $A\#k[M_G]^*$ -module. To simplify the notation, we will write $A[M_G]^*$ for $A\#k[M_G]^*$. Such a module as an A -module is called *coherently G -invariant in this context*. Under this definition, we also have the notion of (strong) G -root as in the finite group case.

Let Q be a finite quiver without oriented cycles. The condition of no oriented cycles is not essential. But otherwise, we need to work with locally finite actions. Since kQ has only finitely many idempotents but G is infinite and connected, G has to fix each idempotent, and thus stabilizes kQ_1 the linear span of arrows. From now on, we assume that G admits a graded polynomial representation theory.

3.1. Description of Q_G . It turns out that $kQ[M_G]^*$ is Morita equivalent to some hereditary algebra kQ_G . The description is completely analogous to the one in 1.1, except that Q_G is possibly an infinite quiver.

Let $\text{irr}(G)$ be the set of all polynomial representations of G , and $R_{uv} := kQ(u, v)$ be the G -module spanned by the arrows from u to v . The vertex set of Q_G is

$$\bigcup_{u \in Q_0} \{u\} \times \text{irr}(G).$$

The arrow set from (u, ρ) to (v, σ) is a basis of

$$\text{Hom}_G(V_\rho, R_{uv} \otimes V_\sigma).$$

Theorem 3.3. *$kQ[M_G]^*$ is Morita equivalent to the path algebra kQ_G .*

Proof. Let R be the (maximal semisimple) subalgebra of kQ generated by the primitive idempotents, and $R_1 \subset kQ$ be the R -bimodule spanned by the arrows, so kQ is the tensor algebra $T(R, R_1)$.

We fix for each $u \in Q_0$ and $\rho \in \text{irr}(G)$, a primitive idempotent e_ρ of $k[M_G]^*$ corresponding to ρ (see Lemma 2.5). Then $\{e_u \otimes e_\rho\}_{u \in Q_0, \rho \in \text{irr}(G)}$ is a basic set of primitive orthogonal idempotents of $kQ[M_G]^*$. Let $e = \sum_{u \in Q_0, \rho \in \text{irr}(G)} e_u \otimes e_\rho$, then

$$eR[M_G]^*e = \prod_{u \in Q_0, \rho \in \text{irr}(G)} ke_u \otimes e_\rho.$$

As G stabilizes R and R_1 , it is easy to see that we have equivalence of categories $\text{mod } kQ[M_G]^* \cong \text{mod } T(R[M_G]^*, R_1[M_G]^*) \cong \text{mod } T(eR[M_G]^*e, eR_1[M_G]^*e)$.

$$\begin{aligned} e_u \otimes e_\rho (R_1[M_G]^*) e_v \otimes e_\sigma &= e_u \otimes e_\rho (R_1 e_v \otimes k[M_G]^* e_\sigma) \\ &= e_\rho (R_{uv}[M_G]^* e_\sigma) \\ &= \text{Hom}_k(k, e_\rho(k[M_G]^*))(R_{uv}[M_G]^* e_\sigma) \\ &= \text{Hom}_G(V_\rho, (R_{uv}[M_G]^* e_\sigma)) \\ &= \text{Hom}_G(V_\rho, R_{uv} \otimes V_\sigma). \end{aligned}$$

□

Since we are mainly interested in coherently G -indecomposable and indecomposable G -invariant representations, it is enough to focus on connected components of Q_G .

Proposition 3.4. *If the quiver Q is finite without oriented cycles, then each connected component of Q_G is finite without oriented cycles.*

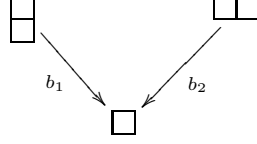
Proof. Since Q is finite, the linear span of arrows is a G -module of bounded degree. So for each $e_u \otimes e_\rho$, it can be connected to only finitely many $e_v \otimes e_\sigma$. But Q has finitely many vertices, the component containing $e_u \otimes e_\rho$ must be finite as well. Since Q has no oriented cycles, we can totally order the vertices of Q such that $u < v$ if there is an arrow $u \rightarrow v$. Now for a given component Q_c , we can totally order the vertices in Q_c by $(u, \rho) < (v, \sigma)$ if $u < v$. Note that $u < v$ is a necessary condition for there is an arrow $(u, \rho) \rightarrow (v, \sigma)$. □

We fix a connected component Q_c of Q_G . Let $A := kQ$ and $B := kQ_c$. Let $T_c : \text{mod } A \rightarrow \text{mod } B$ be the functor $e(A[M_G]^* \otimes_A -)$ followed by the restriction to Q_c . Let $R_c : \text{mod } B \rightarrow \text{mod } A$ be the functor $\text{Hom}_{Q_c}(eA[M_G]^*, -)$ followed by the restriction to A . It is right adjoint to T_c , and can be lifted to an algebraic morphism $\text{Rep}_\beta(Q_c) \rightarrow \text{Rep}_{r_c(\beta)}(Q)$ using a method similar to that in 1.1.

Example 3.5. For each finite quiver Q , we can associate a torus $T_1 = (k^*)^{Q_1}$ acting naturally on kQ_1 . The irreducible representations of T_1 are all one-dimensional indexed by the weight lattice \mathbb{Z}^{Q_1} . So the quiver Q_G from our recipe is the universal abelian covering quiver of Q .

Example 3.6. Let K_n be the n -arrow Kronecker quiver. The general linear group GL_n acts naturally on the arrow space of K_n . This induces an action of GL_n on kK_n . The dimension of $\text{Hom}_G(V_\rho, R_{uv} \otimes V_\sigma)$ is equal to the Littlewood-Richardson coefficients $c_{\sigma,1}^\rho$.

For any $n \geq 2$, the first component of Q_G is always the following quiver.



We can easily compute the functor R_c using Example 2.6. For $n = 3$, The functor R_c takes a representation of the above quiver to the following representation of K_3 .

$$A_1 = \begin{pmatrix} 0 & -B_1 & 0 \\ 0 & 0 & B_1 \\ 0 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -B_1 \\ B_2 & 0 & 0 \\ 0 & 0 & B_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 & 0 \\ -B_1 & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & 0 \\ B_2 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_2 \end{pmatrix}.$$

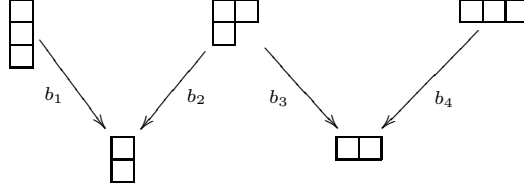
We observed that as the above situation, the matrices obtained are quite sparse. So we will use Yale form to represent them. For example, the A_1 above is the block matrix $\begin{pmatrix} A_{1u} \\ A_{1d} \end{pmatrix}$, where $A_{1u} = \begin{bmatrix} -1 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} B_1$ and $A_{1d} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 1 \end{bmatrix} B_2$.

For $n = 4$, the functor R_c takes a representation of the above quiver to the representation $A_i = \begin{pmatrix} A_{iu} \\ A_{id} \end{pmatrix}$ of K_4 , where

$$A_{1u} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 4 \end{bmatrix} B_1, \quad A_{2u} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 5 \\ 1 & 3 & 4 \end{bmatrix} B_1, \quad A_{3u} = \begin{bmatrix} -1 & 1 & 1 \\ 2 & 3 & 6 \\ 1 & 2 & 4 \end{bmatrix} B_1, \quad A_{4u} = \begin{bmatrix} -1 & 1 & -1 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} B_1;$$

$$A_{1d} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 7 \\ 2 & 3 & 4 & 1 \end{bmatrix} B_2, \quad A_{2d} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 8 \\ 1 & 3 & 4 & 2 \end{bmatrix} B_2, \quad A_{3d} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 6 & 9 \\ 1 & 2 & 4 & 3 \end{bmatrix} B_2, \quad A_{4d} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 5 & 6 & 10 \\ 1 & 2 & 3 & 4 \end{bmatrix} B_2.$$

Example 3.7. For $n \geq 3$, the second component of Q_G is the following quiver



Using Example 2.7, we find that for $n = 3$, the functor R_c takes a representation of the above quiver to the following representation of K_3 .

$$A_i = \begin{pmatrix} A_{iu} & 0 \\ A_{il} & A_{ir} \\ 0 & A_{id} \end{pmatrix} \quad \text{where}$$

$$A_{1u} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} B_1, \quad A_{2u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} B_1, \quad A_{3u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} B_1;$$

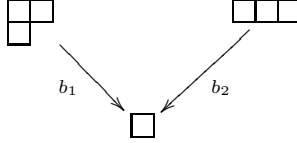
$$A_{1l} = \begin{bmatrix} -2 & 2 & -1 \\ 1 & 3 & 7 \\ 1 & 2 & 3 \end{bmatrix} B_2, \quad A_{2l} = \begin{bmatrix} 2 & 2 & 1 & -1 \\ 2 & 4 & 7 & 8 \\ 1 & 3 & 2 & 2 \end{bmatrix} B_2, \quad A_{3l} = \begin{bmatrix} 2 & 2 & 1 \\ 5 & 6 & 8 \\ 2 & 3 & 1 \end{bmatrix} B_2;$$

$$A_{1r} = \begin{bmatrix} 1 & -2 & 1 & 2 & \frac{1}{2} & -1 \\ 1 & 2 & 3 & 5 & 7 & 8 \\ 1 & 5 & 2 & 6 & 3 & 3 \end{bmatrix} B_3, \quad A_{2r} = \begin{bmatrix} -2 & 1 & -1 & -2 & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & 4 & 6 & 7 & 8 \\ 4 & 1 & 3 & 6 & 2 & 2 \end{bmatrix} B_3, \quad A_{3r} = \begin{bmatrix} -2 & 2 & -1 & 1 & -1 & \frac{1}{2} \\ 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 2 & 3 & 1 & 1 \end{bmatrix} B_3;$$

$$A_{1d} = \begin{bmatrix} 3 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{8} \\ 1 & 4 & 5 & 6 & 8 & 10 \\ 4 & 1 & 5 & 2 & 6 & 3 \end{bmatrix} B_4, \quad A_{2d} = \begin{bmatrix} 3 & 1 & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \\ 2 & 4 & 5 & 7 & 9 & 10 \\ 5 & 4 & 1 & 3 & 6 & 2 \end{bmatrix} B_4, \quad A_{3d} = \begin{bmatrix} 3 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \\ 3 & 6 & 7 & 8 & 9 & 10 \\ 6 & 4 & 5 & 2 & 3 & 1 \end{bmatrix} B_4.$$

The next connect component is a Dynkin- E_7 for $n = 3$ and extended- E_7 for $n > 3$. Other components are all wild quivers.

Example 3.8. As our last example, we still take the quiver K_3 but with a different action. We assume that the 3-dimensional spaces of arrows is the GL_2 -module $S^2(k^2)$. Then the first component of Q_G is



The functor R_c takes a representation of the above quiver to the following representation of K_3 .

$$A_1 = \begin{pmatrix} 0 & -B_1 \\ 0 & 0 \\ 3B_2 & 0 \\ 0 & 0 \\ 0 & B_2 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} B_1 & 0 \\ 0 & B_1 \\ 0 & 0 \\ 2B_2 & 0 \\ 0 & 2B_2 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 \\ -B_1 & 0 \\ 0 & 0 \\ 0 & 3B_2 \\ 0 & 0 \\ B_2 & 0 \end{pmatrix}.$$

4. APPLICATION TO TENSOR INVARIANTS

Let us briefly recall Schofield's semi-invariants of quiver representations [14]. For a fixed dimension vector α , the space of all α -dimensional representations is

$$\mathrm{Rep}_\alpha(Q) := \bigoplus_{a \in Q_1} \mathrm{Hom}(k^{\alpha(ta)}, k^{\alpha(ha)}).$$

The product of general linear group $\mathrm{GL}_\alpha := \prod_{v \in Q_0} \mathrm{GL}_{\alpha(v)}$ acts on $\mathrm{Rep}_\alpha(Q)$ by the natural base change. This action has a *kernel*, which is the multi-diagonally embedded k^* . For any *weight* $\sigma \in \mathbb{Z}^{Q_0}$, we can associate a character of GL_α still denoted by σ

$$(g(v))_{v \in Q_0} \mapsto \prod_{v \in Q_0} (\det g(v))^{\sigma(v)}.$$

We define the subgroup $\mathrm{GL}_\alpha^\sigma$ to be the kernel of the character map. The semi-invariant ring $\mathrm{SIR}_\alpha^\sigma(Q) := k[\mathrm{Rep}_\alpha(Q)]^{\mathrm{GL}_\alpha^\sigma}$ of weight σ is σ -graded: $\bigoplus_{n \geq 0} \mathrm{SI}_\alpha^{n\sigma}(Q)$, where

$$\mathrm{SI}_\alpha^\sigma(Q) := \{f \in k[\mathrm{Rep}_\alpha(Q)] \mid g(f) = \sigma(g)f, \forall g \in \mathrm{GL}_\alpha\}.$$

For any $N \in \mathrm{Rep}_\beta(Q)$, we take some injective resolution of N

$$0 \rightarrow N \rightarrow I_0 \rightarrow I_1 \rightarrow 0,$$

and apply the functor $\mathrm{Hom}_Q(M, -)$ for $M \in \mathrm{Rep}_\alpha(Q)$

$$(4.1) \quad \mathrm{Hom}_Q(M, N) \hookrightarrow \mathrm{Hom}_Q(M, I_0) \xrightarrow{\phi_M^N} \mathrm{Hom}_Q(M, I_1) \twoheadrightarrow \mathrm{Ext}_Q(M, N).$$

If $\langle \alpha, \beta \rangle_Q = 0$, then ϕ_M^N is a square matrix. Following Schofield [14], we define $c(M, N) := \det \phi_M^N$. It is not hard to see that the definition does not depend on the injective resolution of N . In particular, we can take the canonical resolution or minimal resolution of N . We can also define $c(M, N)$ using projective resolution of M . Note that $c(M, N) \neq 0$ if and only if $\mathrm{Hom}_Q(M, N) = 0$ or $\mathrm{Ext}_Q(M, N) = 0$. We denote $c_N := c(-, N)$ and dually $c^M := c(M, -)$.

It is proved in [14] that $c_N \in \text{SI}_\alpha^{\sigma_\beta^\vee}(Q)$ for $\sigma_\beta^\vee = -\langle -, \beta \rangle_Q$, and dually $c^M \in \text{SI}_\beta^{\sigma_\alpha}(Q)$ for $\sigma_\alpha = \langle \alpha, - \rangle_Q$. In fact, c_N 's (resp. c^M 's) span $\text{SI}_\alpha^{\sigma_\beta^\vee}(Q)$ (resp. $\text{SI}_\beta^{\sigma_\alpha}(Q)$) over the base field k [2, 15, Theorem 1].

Let G be a finite group or an infinite connected reductive group acting polynomially on kQ as automorphisms. Such an action induces a rational action of G on all representation spaces of Q . We are interested in those semi-invariants which is also semi-invariant under the G -action.

Observation 4.1. *If N is proj-coherently G -invariant, then c_N is also semi-invariant under G -action.*

Proof. Since N is proj-coherently G -invariant, there is some map $\varphi : G \rightarrow \text{GL}_\alpha$ such that $N^g = \varphi(g)N$ and φ descends to a representation $G \rightarrow \text{GL}_\alpha/k^*$. Then

$$c_{N^g}(M) = c^M(\varphi(g)N) = \sigma_\alpha(\varphi(g))c^M(N) = (\sigma_\alpha\varphi)(g)c_N(M).$$

Since $\langle \alpha, \beta \rangle_Q = 0$, $\sigma_\alpha|_{k^*}$ is trivial, so $\sigma_\alpha\varphi$ is a character of G . In other words c_M is semi-invariant under G -action. \square

This observation allows us to construct a lot of new semi-invariants for the $\text{GL}_\alpha \times G$ -action on $k[\text{Rep}_\alpha(Q)]$. According to Observation 1.4, any exceptional (=rigid Schur) representation is proj-coherently G -invariant. Actually we conjecture that they are all coherently G -invariant. The dimension of such a representation is a *real Schur root* γ of the quiver. Moreover, for any two general representations $N_1, N_2 \in \text{Rep}_\gamma(Q)$, c_{N_1} is a multiple of c_{N_2} . In this sense, we will treat these semi-invariants as trivial, and avoid them later.

We are particularly interested in applying the method to construct the semi-invariants of (tri)-tensors. By a (tri)-tensor of vector spaces (U, V, W) , we mean the vector space $U^* \otimes V \otimes W^*$. The product of special linear groups $SL := \text{SL}(U) \times \text{SL}(V) \times \text{SL}(W)$ acts naturally on it. We are interested in the invariants in $k[U^* \otimes V \otimes W^*]$ for this action. The tensor space can be identified with the (α_1, α_2) -dimensional representation space of the n -arrow Kronecker quiver K_n , where $\dim U = \alpha_1$, $\dim V = \alpha_2$, and $\dim W = n$. In this context, $G = \text{GL}(W)$.

It follows from Example 3.6 that

Proposition 4.2. *For general square matrices B_1, B_2 , we define the representations N_1, N_2 of K_3*

$$\begin{aligned} N_1(a_1) &= \begin{pmatrix} 0 & -B_1 & 0 \\ 0 & 0 & B_1 \\ 0 & 0 & 0 \end{pmatrix}, & N_1(a_2) &= \begin{pmatrix} B_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -B_1 \end{pmatrix}, & N_1(a_3) &= \begin{pmatrix} 0 & 0 & 0 \\ -B_1 & 0 & 0 \\ 0 & B_1 & 0 \end{pmatrix}, \\ N_2(a_1) &= \begin{pmatrix} 0 & B_2 & 0 \\ 0 & 0 & B_2 \\ B_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & N_2(a_2) &= \begin{pmatrix} B_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_2 \\ 0 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & N_2(a_3) &= \begin{pmatrix} 0 & 0 & 0 \\ B_2 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_2 \end{pmatrix}. \end{aligned}$$

Then c_{N_1} (resp. c_{N_2}) is a semi-invariant for the tensor of size $a \times 2a \times 3$ (resp. $a \times a \times 3$).

Proposition 4.3. *For general square matrices B_1, B_2 , we define the representations N_1, N_2 of K_4 by A_{iu}, A_{id} as in Example 3.6, then c_{N_1} (resp. c_{N_2}) is a semi-invariant for the tensor of size $2a \times 5a \times 4$ (resp. $2a \times 3a \times 4$).*

Proposition 4.4. *For general square matrices B_1, B_2, B_3, B_4 , we define the representations N_1, N_2 of K_3 by A_{ir}, A_{id} as in Example 3.7, then c_{N_1} (resp. c_{N_2}) is a semi-invariant for the tensor of size $3a \times 5a \times 3$ (resp. $3a \times 4a \times 3$).*

We define the representation N_3, N_4, N_5 of K_3 by

$$N_3(a_i) = \begin{pmatrix} A_{il} & A_{ir} \end{pmatrix}, N_4(a_i) = \begin{pmatrix} A_{il} & A_{ir} \\ 0 & A_{id} \end{pmatrix}, N_5(a_i) = \begin{pmatrix} A_{iu} & 0 \\ A_{il} & A_{ir} \\ 0 & A_{id} \end{pmatrix}.$$

then c_{N_3} (resp. c_{N_4}, c_{N_5}) is a semi-invariant for the tensor of size $9a \times 19a \times 3$ (resp. $8a \times 9a \times 3, a \times a \times 3$).

We remark that our construction also applies to the case when the third factor W is other representation of $\mathrm{GL}(W)$.

Proposition 4.5. *For general square matrices B_1, B_2 , we define the representations N_1, N_2 of K_3 (see Example 3.8)*

$$\begin{aligned} N_1(a_1) &= \begin{pmatrix} 0 & B_1 \\ 0 & 0 \end{pmatrix}, & N_1(a_2) &= \begin{pmatrix} B_1 & 0 \\ 0 & B_1 \end{pmatrix}, & N_1(a_3) &= \begin{pmatrix} 0 & 0 \\ -B_1 & 0 \end{pmatrix}, \\ N_2(a_1) &= \begin{pmatrix} 3B_2 & 0 \\ 0 & B_2 \\ 0 & 0 \end{pmatrix}, & N_2(a_2) &= \begin{pmatrix} 0 & 0 \\ 2B_2 & 0 \\ 0 & 2B_2 \end{pmatrix}, & N_2(a_3) &= \begin{pmatrix} 0 & 0 \\ 0 & 3B_2 \\ B_2 & 0 \end{pmatrix}. \end{aligned}$$

Then c_{N_1} (resp. c_{N_2}) is a semi-invariant in $k[U^* \otimes V \otimes S^2(W)^*]$ for $\dim(U, V, W) = (a, 2a, 2)$ (resp. $(a, a, 2)$).

Fix a component Q_c of Q_G . Let $\mathrm{SI}_\alpha^{\sigma_{R_c(\beta)}^\vee}(Q)$ be the vector space spanned by semi-invariants on $\mathrm{Rep}_\alpha(Q)$ of form $c_{R_c(N)}$ for $N \in \mathrm{Rep}_\beta(Q_c)$. On the other hand, we can restrict a semi-invariant $c_N \in \mathrm{SI}_{r_c(\beta)}^{\sigma_\alpha}(Q)$ on the subvariety $R_c(\mathrm{Rep}_\beta(Q_c))$. We denote the linear span of these restricted semi-invariants by $\mathrm{SI}_{R_c(\beta)}^{\sigma_\alpha}(Q)$. Similar to [2, Corollary 1], we have the following reciprocity property

Proposition 4.6. $\dim \mathrm{SI}_\alpha^{\sigma_{R_c(\beta)}^\vee}(Q) = \dim \mathrm{SI}_{R_c(\beta)}^{\sigma_\alpha}(Q)$.

In general, we do not know a simple method to compute the dimension of $\mathrm{SI}_\alpha^{\sigma_{R_c(\beta)}^\vee}(Q)$. Sometimes, it is easier to perform computation on Q_c using the theorem below. To prove the theorem, we need some construction related to the functor T_c . We can algebraically lift T_c as we did for R_c . Moreover, the lifting can be constructed at the level of morphisms. For our purpose, we only state such a lifting for morphisms between projectives. It is enough to do this for $P_v \xrightarrow{a} P_u$, where P_u, P_v are indecomposable projective representations corresponding to $u, v \in Q_0$, and a is an arrow $u \rightarrow v$. The construction will depend on the lifting of R_c . Recall that a lifting of R_c maps a representation N of Q_c to a representation M of Q as follows. The vector space M_u attached to the vertex u is $M_u = \bigoplus_{\rho \in Q_c} \dim(V_\rho) N_{u\rho}$. Here, by $\rho \in Q_c$ we mean that there is an idempotent in Q_c corresponding to the irreducible representation ρ . The linear map from the i -th copy of $N_{u\rho}$ to j -th copy of $N_{v\sigma}$ is given by substituting the arrows b_k in certain linear combination $\sum_k c_k^{ij} b_k$ by corresponding matrices in N .

Now we let T_c send P_u to $T_c(P_u) = \bigoplus_{\rho \in Q_c} \dim(V_\rho) P_{u\rho}$, and send the morphism $P_v \xrightarrow{a} P_u$ to a matrix with $\sum_k c_k^{ij} b_k$ as the ij -th entry. We see from the construction that such a lifting is not only algebraic but also compatible with the adjunction in the sense that $\mathrm{Hom}_Q(P_u, R_c(N))$ can be naturally identified with

$\text{Hom}_{Q_c}(T_c(P_u), N)$ such that the diagram commutes

$$\begin{array}{ccc} \text{Hom}_Q(P_u, R_c(N)) & \xrightarrow{\text{Hom}_Q(a, R_c(N))} & \text{Hom}_Q(P_v, R_c(N)) \\ \parallel & & \parallel \\ \text{Hom}_{Q_c}(T_c(P_u), N) & \xrightarrow{\text{Hom}_{Q_c}(T_c(a), N)} & \text{Hom}_{Q_c}(T_c(P_v), N). \end{array}$$

We remind readers that a morphism $P_1 \xrightarrow{f} P_0$ can be represented by a matrix whose entries are linear combination of pathes, and apply $\text{Hom}_Q(-, N)$ to this morphism is nothing but substitute arrows in the matrix by corresponding matrix representation in N .

Let $\text{SI}_{\beta}^{\sigma_{T_c(\alpha)}}(Q_c)$ be the vector space spanned by semi-invariants on $\text{Rep}_{\beta}(Q_c)$ of form $c^{T_c(M)}$ for $M \in \text{Rep}_{\alpha}(Q)$.

Theorem 4.7. $\dim \text{SI}_{\alpha}^{\sigma_{R_c(\beta)}}(Q) = \dim \text{SI}_{\beta}^{\sigma_{T_c(\alpha)}}(Q_c)$.

Proof. For any two representations $M \in \text{Rep}_{\alpha}(Q), N \in \text{Rep}_{\beta}(Q_c)$, we take the canonical resolution $0 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$, and apply the functor $\text{Hom}_Q(-, R_c(N))$, then we get

$$\begin{array}{ccccccc} \text{Hom}_Q(M, R_c(N)) & \hookrightarrow & \text{Hom}_Q(P_0, R_c(N)) & \xrightarrow{\phi_M^{R_c(N)}} & \text{Hom}_Q(P_1, R_c(N)) & \twoheadrightarrow & \text{Ext}_Q(M, R_c(N)) \\ \parallel & & \parallel & & \parallel & & \parallel \\ \text{Hom}_{Q_c}(T_c(M), N) & \hookrightarrow & \text{Hom}_{Q_c}(T_c(P_0), N) & \xrightarrow{\phi_{T_c(M)}^N} & \text{Hom}_{Q_c}(T_c(P_1), N) & \twoheadrightarrow & \text{Ext}_{Q_c}(T_c(M), N). \end{array}$$

The lower row is due to the adjunction. Since T_c is exact and preserve projectives, $0 \rightarrow T_c(P_1) \rightarrow T_c(P_0) \rightarrow T_c(M) \rightarrow 0$ is in fact a projective resolution of $T_c(M)$. By our construction of T_c , we conclude that

$$c(M, R_c(N)) = \det \phi_M^{R_c(N)} = \det \phi_{T_c(M)}^N = c(T_c(M), N).$$

Therefore, $\dim \text{SI}_{\alpha}^{\sigma_{R_c(\beta)}}(Q) = \dim \text{SI}_{\beta}^{\sigma_{T_c(\alpha)}}(Q_c)$. \square

As an example, let us compute the dimension of $\text{SI}_{(1,2)}^{\sigma_{R_c(1,0,1)}}(K_3)$ in Proposition 4.2. It is enough to compute the dimension of $\text{SI}_{(1,0,1)}^{\sigma_{T_c(1,2)}}(Q_c)$. This Q_c is a finite type quiver, so the dimension of $\text{SI}_{(1,0,1)}^{\sigma_{T_c(1,2)}}(Q_c)$ is at most one. A general representation M in $\text{Rep}_{(1,2)}(K_3)$ has resolution $0 \rightarrow P_2 \xrightarrow{k_1 a_1 + k_2 a_2 + k_3 a_3} P_1 \rightarrow M \rightarrow 0$, then

$$0 \rightarrow T_c(P_2) = 3P_{\square} \xrightarrow{\begin{pmatrix} k_2 b_1 & -k_1 b_1 & 0 \\ -k_3 b_1 & 0 & k_1 b_1 \\ 0 & k_3 b_1 & -k_2 b_1 \end{pmatrix}} 3P_{\blacksquare} = T_c(P_1) \rightarrow T_c(M) \rightarrow 0.$$

Now it is not hard to see that $T_c(M)$ decomposes as $3(M_1 \oplus M_2)$, where M_1 (resp. M_2) is a general representation of dimension $(0, 1, 1)$ (resp. $(1, 1, 1)$). So we see that $\text{Hom}_{Q_c}(T_c(M), N) = 0$ for general $N \in \text{Rep}_{(1,0,1)}(Q_c)$, and thus $\dim \text{SI}_{(1,2)}^{\sigma_{R_c(1,0,1)}}(K_3) = 1$. In fact, $\dim \text{SI}_{(a,2a)}^{\sigma_{R_c(1,0,1)}}(K_3) = 1$ for all $a \in \mathbb{N}$.

We checked that the space of semi-invariants of fixed weight in Proposition 4.2, 4.3, 4.4, and 4.5 are all one-dimensional by hand and by computer. This theorem

also tells us that to construct nontrivial semi-invariants, it is enough to use those *stable* representation of Q_c in the sense of [8].

ACKNOWLEDGEMENT

The author wants to thank Liping Li for introducing him to think in *EI*-categories. He would also like to thank Professor Doty for some discussion on Schur algebras.

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